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EFFECTS LIMITING ACCELERATED BEAM INTENSITY IN THE LARGEST PROTON SYNCHROTRON

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In this review, we will discuss the two major categories of beam-dynamical considerations in designing very large proton synchrotrons that bear direct relation to the ultimate beam intensity achievable. To begin with, we take for granted that in all future multi-TeV proton synchrotrons, limitations of physical size and electric-power consumption will necessitate the use of superconducting magnets. The relatively large field errors encountered in these magnets demand a closer re-examination of single-particle dynamics. In addition, the extensively investigated self-field effects will be present and will be equally important at beam currents similar to those in smaller accelerators. A third major consideration that may also limit the attainable intensity is the effect of stray beam striking superconducting magnets and causing them to quench. Very little quantitative information is available. Until more experience with superconducting magnets used in accelerators is gained, this will remain the most serious concern in the design of superconducting synchrotron. Nevertheless, since this is not primarily a beam-dynamics problem, we will not discuss it here.

A. Single-Particle Dynamics

In superconducting magnets, the field profile is determined by locations of currents. Precise placement of current-carrying conductors is difficult and the large magnetic forces tend to displace the conductors during pulsing. Therefore the field errors tend to be larger than those in conventional magnets. The effort is generally to attempt reducing errors not to below tolerance, but to a sufficiently low level that corrections are reasonable.

Low-order resonances and distortions of orbit characteristics can always be corrected by distributed correction magnets that are individually adjustable. In the following, we will look at the collective effects of all high-order resonances and distortions, which are difficult if not impossible to correct and which do not decrease sharply with increasing order in superconducting magnets as in conventional magnets.

1. Central Orbit

To get a rough concept of the order of magnitude and the parametric dependence of the effects of high-order resonances, it is adequate to consider only a one-dimensional example. The ring is assumed to be composed of N magnets of different types each of length ℓ (possibly all different) and having a uniform field error represented by $B^{(n)}$ = n^{th} derivative of error field. Errors in different magnets are random and uncorrected. The full width of the $\nu = m/n$ resonance caused by the lowest-order multipole (2n-pole) error is given by standard theory to be

$$(\delta\nu)_{\frac{m}{n}} = \frac{1}{2\pi}\sqrt{\Sigma\left[\frac{\beta}{(n-1)!}\,\frac{B^{(n-1)}\ell}{B\rho}\left(\frac{\beta}{4}\,\frac{\varepsilon}{\pi}\right)^{\frac{n}{2}-1}\right]^2}$$

$$= \frac{\sqrt{\Sigma\ell^2}}{2\pi\rho}\,\frac{\beta}{(n-1)!}\,\frac{B^{(n-1)}}{B}\left(\frac{\beta}{4}\,\frac{\varepsilon}{\pi}\right)^{\frac{n}{2}-1} \tag{1}$$

where β is the linear betatron amplitude function, $B\rho$ is the rigidity of the particle, ε is the emittance, and the summation is over all magnets in the ring. In the last expression β and $B^{(n-1)}$ represent a weighted average of the absolute values of these quantities, lying somewhere between the maximum and the minimum. To obtain this simplified expression, we have also assumed that the phase-space area enclosed by a linear trajectory passing through the unstable fixed points is roughly equal to that bounded by the separatrices. This approximation is quite good for high-order resonances. We will use approximation rather freely, keeping only order-of-magnitude accuracy while attempting to derive some simple physical interpretation and feeling.

To further simplify the expression we note first that if all magnets were of the same length, $\sqrt{\Sigma\ell^2} = \sqrt{N}\,\ell = \frac{N\ell}{\sqrt{N}} \cong \frac{2\pi\rho}{\sqrt{N}}$. This approximation is not too bad even if the magnets are different in length. We also have $\sqrt{\beta\frac{\varepsilon}{\pi}} = a$, the beam half-width. Altogether, we can rewrite Eq. (1) as

$$(\delta\nu)_{\frac{m}{n}} \cong \frac{1}{\sqrt{N}}\,\frac{\beta}{(n-1)!}\,\frac{B^{(n-1)}}{B}\left(\frac{a}{2}\right)^{n-2}$$

$$= \frac{1}{\sqrt{N}}\,\frac{1}{n-1}\,\frac{\beta\delta B'_n}{B} \tag{2}$$

where $\delta B_n' \equiv \frac{1}{(n-2)!} B^{(n-1)} (\frac{a}{2})^{n-2}$. Now we can state Eq. (2) as "The width of an nth order resonance caused by a random 2n-pole error is proportional to the ratio of an indicial field to the bending field. The indicial field is obtained by following the error field gradient roughly halfway out from the beam center to a distance β."

To assess the collective effects of all the high-order resonances, we adopt a criterion similar to that used by Chirikov/1/ for defining the stochasticity limit, namely the sum of the widths of all resonances within a unit tune interval must be smaller than unity. Following Chirikov, we shall also ignore the overlapping of resonances and treat them as though they are all spread out. Omitting the one at integer tune, the number of nth-order resonances in unit tune interval is n-1 and the sum of all the widths is

$$W \equiv \sum_{n=2}^{\infty} (n-1)(\delta\nu)_{\frac{m}{n}} = \frac{1}{\sqrt{N}} \Sigma \frac{\beta}{(n-2)!} \frac{B^{(n-1)}}{B} (\frac{a}{2})^{n-2}$$

$$= \frac{1}{\sqrt{N}} \frac{\beta \Sigma \delta B_n'}{B} \tag{3}$$

Note that in the second expression $\Sigma\ \delta B_n'$ is not the algebraic sum and is hence not just the physical error-field gradient halfway out from beam center. It is rather the sum of (certain weighted average of) the absolute value of field gradients from each 2n-pole error. Therefore, a superposition of several multipole errors giving zero gradient at halfway out from beam center does not give W = 0.

We can now use Eq. (3) to evaluate the effect of errors in a superconducting magnet ring. For this we take a simplistic structure of a superconducting magnet. In cross-section, we assume the field to be produced by the distribution of a single layer of thin conductors on a circle of radius r. The density of conductors is arranged to be $\propto \cos\theta$ for dipole, $\propto \cos 2\theta$ for quadrupole, $\propto \cos 3\theta$ for sextupole etc. The deviation between this quantized current distribution together with rather large unavoidable positioning errors and the ideal smooth distribution can be represented by a large number of δ-functions distributed around the circle. These error δ-function currents, when Fourier analyzed, will contain all components cos $n\theta$ with roughly equal amplitudes. An error current I_n cos $n\theta$ gives

$$B^{(n-1)} = 2\pi\ I_n \frac{(n-1)!}{r^{n-1}}$$

or

$$\frac{B^{(n-1)}}{B} = \frac{I_n}{I_1} \frac{(n-1)!}{r^{n-1}}$$

where $B = 2\pi I_1$ is the ideal field in the bending dipoles. We see here that compared with a conventional magnet, the multipole error field decreases much more slowly with increasing multipole order in a superconducting magnet. Substituting $B^{(n-1)}/B$ in Eq. (3) we get

$$W = \frac{1}{\sqrt{N}} \sum_{n=2}^{\infty} (n-1) \frac{\beta}{r} \frac{I_n}{I_1} \left(\frac{a}{2r}\right)^{n-2} \tag{4}$$

Putting all $I_n = I_e$ = same for all n, we can simplify Eq. (4) to

$$W = \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \sum_{n=2}^{\infty} (n-1) \left(\frac{a}{2r}\right)^{n-2}$$

$$= \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \frac{1}{\left(1-\frac{a}{2r}\right)^2} .$$

The appearance of the term $\frac{a}{2r}$ in the denominator is puzzling. One would expect W to go to ∞ as a approaches r. This discrepancy arises because the resonance width $(\delta\nu)_{m/n}$ as given by Eq. (1) is the contribution from only the lowest order 2n-pole, whereas higher-order poles also contribute to this width. The factor 1/2 disappears when all the contributions are properly included. The derivation will become clear later when we discuss the effects of high-order resonances on off-center orbits. For now we will simply put down the correct expression

$$W = \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \frac{1}{\left(1-\frac{a}{r}\right)^2} . \tag{5}$$

We can now make a few observations:

a. The usual design scaling laws with respect to the particle momentum p for fixed bending magnetic field and length

$$\rho,\ N \propto p$$

$$\nu,\ \beta \propto p^{1/2}$$

$$a,\ r \propto p^{-1/4} \quad (\text{because } \varepsilon \propto p^{-1})$$

do not apply when field errors are considered. Instead, the scaling of r should be

$$r \propto p^{o},$$

namely as energy increases the aperture of the magnets should remain fixed.

b. The ratio $\frac{a}{r}$ should be $< \frac{1}{2}$ so that the denominator does not do much harm. For the Fermilab Energy Doubler/Saver with parameters $r \cong 4$ cm, $\beta \cong 70$ m, $N \cong 800$ and $\frac{a}{r} \cong$ small, or for any synchrotron that scales from these parameters, we get

$$W \cong 60 \frac{I_e}{I_1} .$$

For the superconducting UNK synchrotron, $r \cong 5$ cm, $\beta \cong 100$ m, $N \cong 2000$, $\frac{a}{r} \cong$ small, we get

$$W \cong 45 \frac{I_e}{I_1} .$$

The stochasticity limit corresponds to $W = 1$. But presumably even at an order of magnitude below the stochasticity limit, the beam life time is already limited by Arnol'd diffusion. Thus for both accelerators a safe limit would be

$$\frac{I_e}{I_1} < 10^{-3}$$

which is not too difficult to achieve.

c. The simplistic conductor arrangement assumed is quite unrealistic. Most superconducting magnets have conductors arranged in blocks or shells with dimensions so adjusted as to eliminate I_n for low values of n. For these magnets, I_n may be rather large for certain sequences of high n values, extending effectively to $n = \infty$. These conductor arrangements should be examined closely in reference to the criterion given above.

2. Off-center orbits

To take full advantage of the power-saving feature of superconducting magnets, the synchrotron should be cycled slowly with long flattops for long beam spills, despite the fact that some magnets with NbTi conductors can be ramped at a rate higher than 1 T/sec without appreciable degradation of the peak field attainable. At low repetition rate, to obtain reasonably high time-average beam intensity, one should aim to accelerate the largest possible number of protons per pulse. Betatron stacking (multiturn injection) would increase $\frac{a}{r}$ and momentum stacking would require stable containment of beams on off-center orbits. On an orbit at displacement b off center, the multipole field derivatives $B_b^{(n-1)}$ are given by

$$B_b^{(n-1)} = \sum_{k=o}^{\infty} \frac{1}{k!} B^{(n-1+k)} b^k .$$

For superconducting magnets this gives

$$\frac{B_b^{(n-1)}}{B} = \sum_{k=o}^{\infty} \frac{(n-1+k)!}{k!} \frac{I_{n+k}}{I_1} \frac{1}{r^{n-1}} \left(\frac{b}{r}\right)^k$$

and

$$W = \frac{1}{\sqrt{N}} \sum_{n=2}^{\infty} \sum_{k=o}^{\infty} \frac{\beta}{r}(n-1+k) \frac{(n-2+k)!}{(n-2)!\,k!} \frac{I_{n+k}}{I_1} \left(\frac{a}{2r}\right)^{n-2} \left(\frac{b}{r}\right)^k$$

$$= \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \sum_{\ell=o}^{\infty} (\ell+1) \sum_{k=o}^{\infty} \frac{\ell!}{(\ell-k)!\,k!} \left(\frac{a}{2r}\right)^{\ell-k} \left(\frac{b}{r}\right)^k$$

$$= \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \sum_{\ell=o}^{\infty} (\ell+1) \left(\frac{a}{2r} + \frac{b}{r}\right)^{\ell}$$

$$= \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \frac{1}{\left(1 - \frac{a/2+b}{r}\right)^2}$$

In deriving the corrected formula, Eq. (5), a similar procedure is used to take into account all contributions from higher-order multipole errors to resonance widths. The absence of the factor $\frac{1}{2}$ in b suggests the outcome of this procedure which gives, in this case,

$$W = \frac{1}{\sqrt{N}} \frac{I_e}{I_1} \frac{\beta}{r} \frac{1}{\left(1 - \frac{a+b}{r}\right)^2} . \tag{6}$$

Thus if $\frac{a+b}{r} \cong \frac{2}{3}$ the allowable I_e/I_1 must be reduced by about an order of magnitude. With some reinterpretation, it is easy to relate b to closed-orbit distortion. Therefore it is especially important in superconducting synchrotrons to correct closed-orbit distortions carefully.

3. Emittance growth and resonant extraction

Random field errors cause beam emittance to grow. Emittance growth caused by field errors in a synchrotron is negligibly small because the periodic nature (revolution period) of the errors concentrates all their emittance blow-up effects into resonances. Away from resonances, the effects of field errors from successive revolutions add up vectorially to zero. As synchrotrons get larger, the emittance growth during one revolution becomes larger and the delicate cancellation between successive turns becomes less precise. We will make an estimate here of the magnitude of this effect.

The emittance growth caused by a single δ-function angle kick

$$\delta x' = (\delta k)x \equiv \left[\delta\left(\frac{B'\ell}{B\rho}\right)\right]x$$

is

$$\begin{aligned}\delta(2J) &= 2(\alpha x+\beta x')\delta x'+\beta(\delta x')^2\\ &= \Big[-2(\beta\delta k)\sin(\phi+\nu\theta)\ \cos(\phi+\nu\theta)\\ &\qquad +(\beta\delta k)^2\ \cos^2(\phi+\nu\theta)\Big](2J)\end{aligned} \tag{7}$$

where $\theta = \int\frac{ds}{\nu\beta}$ is the betatron phase normalized to 2π per revolution and

$$\begin{cases} J = \dfrac{1}{2}\dfrac{\varepsilon}{\pi} = \dfrac{1}{2}(\gamma x^2+2\alpha x x'+\beta x'^2)\\ \phi = -(\nu\theta + \tan^{-1}\dfrac{\alpha x+\beta x'}{x})\end{cases}$$

are the conjugate variables with the linear betatron motion taken out, so that in an ideal field both ϕ and J are constant. As before, we are identifying the invariant $2J$ of a particle on the beam envelope with the emittance ε/π of the beam. For random δk occuring at random phase θ, averaging over many kicks we get

$$\frac{d(2J)}{dn} = \frac{1}{2}\left\langle(\beta\delta k)^2\right\rangle(2J)$$

or

$$\varepsilon = \varepsilon_o e^{\frac{1}{2}\left\langle(\beta\delta k)^2\right\rangle n} \tag{8}$$

where n is the number of kicks. For a FODO transport with 90^o phase advance per cell, $(\beta k)^2 \cong 8$, and

$$\frac{1}{2}\left\langle(\beta\delta k)^2\right\rangle \cong 4\left\langle\left(\frac{\delta k}{k}\right)^2\right\rangle.$$

In a superconducting magnet ring, the contribution to $\left\langle (\frac{\delta k}{k})^2 \right\rangle$ from the dipoles can easily be as large as 10^{-6}. This gives an e-folding kick number of $n_e = 250000$. For the UNK synchrotron, with n = 360 per turn, this is a mere 700 turns, which means that the beam emittance would increase e-fold in 700 turns if the kicks were not periodic. With strictly periodic δk, it is easy to see that in successive revolutions the kicks due to a given δ-function error are related in amplitude and phase so as to periodically cancel one another out in the summation, but, many effects can destroy the strict turn-periodicity of the field error and cause a secular buildup of the emittance. For example, with phase oscillations superposed (which in large synchrotrons have periods of tens or even hundreds of revolutions), the strict periodicity of δk becomes that of the phase oscillation. In large conventional-magnet synchrotrons, the magnet ripple at low harmonics of 50 or 60 Hz will also lengthen the strict periodicity of δk. This may well be responsible for the much smaller available betatron aperture compared with the momentum aperture that is observed on large conventional synchrotrons such as the Fermilab main ring and the CERN SPS.

During resonant extraction, after coming out of the central stable phase region a section of beam will rapidly grow in coherent oscillation amplitude. The field errors encountered by this section of beam will, therefore, not have strict turn-periodicity. The growth of its emittance, especially that in the perpendicular plane, must be investigated in a similar manner.

B. Beam Self-Field Effects

Several comprehensive review papers have been published on this subject. It is superfluous to review this subject again, especially since no new effect has been discovered since it was last reviewed/2/ in 1975. Furthermore, a most detailed review paper is scheduled for publication in "Particle Accelerators" in the near future. Nevertheless, it may be useful to give a synopsis of the general mathematical formulation and the method of solution for these dynamic self-field phenomena. We will also make some statement of the general behavior of the solution.

These general behaviors are so far not susceptible to analytical derivation. Numerical studies using a computer are the only means of approach. This approach is certainly limited both in accuracy and in generality. Thus, some of the general statements are necessarily extrapolations of information obtained on isolated problems using specific approximations and models.

In the most general form, the beam is described by a distribution function ψ in the 6-dimensional phase space (q_i, p_i), $i = 1,2,3$. To simplify formulas in the following we shall drop the subscript i as being understood and write

$$\psi = \psi(q,p;t) .$$

The time-development of ψ is given by the Vlasov equation

$$\frac{\partial\psi}{\partial t} + \dot{q}\,\frac{\partial\psi}{\partial q} + \dot{p}\,\frac{\partial\psi}{\partial p} = 0 \qquad (9)$$

which is the continuity equation for an incompressible fluid in the phase space. The dynamics of the fluid motion is given by a Hamiltonian H through the canonical equations. With beam-field included, H is also dependent on the spatial distribution $\int\psi dp$ of the beam particles. The charge and current distributions $e\int\psi dp$ and $e\dot{q}\int\psi dp$ give the beam-field through an impedance function Z of the electromagnetic surrounding of the beam. The beam-field then enters H in the same manner as the external field. Thus we can write

$$H = H(q,p,\int\psi dp;t)$$

and

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} .$$

In most cases of interest the Hamiltonian can be transformed to a form that is explicitly independent of t.

The straightforward problem is the following: starting with a given ψ at $t = 0$, solve Eq. (9) for ψ at any later time t. In practice, however, it is more useful to know the various types of solutions the equation yields and their properties. General studies of the solutions of Eq. (9) consist of 3 steps:

1. Find stationary (time-independent) solutions. Evidently, stationary solutions exist only for explicitly time-independent Hamiltonians.

2. Determine the condition of stability for each stationary solution. It is believed that a stable beam in an accelerator would eventually (not necessarily a very long time) settle into one of the stable stationary solutions.

3. For unstable distributions, find their time evolutions.

We shall discuss briefly the method of proceeding for each step.

1. Stationary solutions

For these solutions Eq. (9) becomes

$$\frac{\partial H}{\partial p}\frac{\partial \psi}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \psi}{\partial p} = 0$$

with

$$H = H(q,p,\int \psi dp).$$

Clearly any distribution of the form

$$\psi = \psi(H) \tag{10}$$

will be stationary. Eq. (10) is an integral equation which must be solved for ψ. The most interesting form of Eq. (10) is

$$\psi \propto e^{-\lambda H}. \tag{11}$$

Aside from the dependence through $\int \psi dp$, the lowest-order terms in H are generally quadratic in q and p. Hence Eq. (11) corresponds to Gaussian distributions when the intensity is low. At high intensities, approximate solutions of Eq. (11) have been obtained for the one-dimensional case. For the two-dimensional case, the only known exact solution of Eq. (10) is the Kapchinsky-Vladimirsky distribution/3/ which is a δ-function distribution on the contour line H = const.

These solutions are stationary only for an ideal external field. Turning on field errors will make those particles in the distribution that lie inside resonance bands unstable. This instability is inherent in the Hamiltonian, in contrast to the coherent dynamic instability which arises from the Vlasov equation. The former may be called the Hamilton instability or the particle instability (static, incoherent) and the latter may be called the Vlasov instability or the distribution instability (dynamic, coherent). Thus, one wants to calculate the tune-shifts and tune-spread of a given stationary distribution to see at what intensity the tune would have to cross a strong resonance.

2. Stability

To investigate the stability of a stationary solution $\psi_o(q,p)$, we add a small perturbation $\psi_1(p)e^{i(nq-\omega t)}$ and expand both H and the Vlasov equation to linear terms in ψ_1. Note that we have taken the perturbation to be a single mode in the expansion in traveling-wave modes. A general perturbation would be a superposition of these traveling waves. For the expanded Hamiltonian, we write

$$H = H_o + Ke^{i(nq-\omega t)}\int \psi_1 dp.$$

The linearized Vlasov equation becomes

$$\dot{p}_o \frac{d\psi_1}{dp} + \left[i(n\dot{q}_o - \omega) - K\frac{\partial \psi_o}{\partial q} \right]\psi_1 = \left[\left(\frac{\partial K}{\partial q} + inK \right) \frac{\partial \psi_o}{\partial p} - \frac{\partial K}{\partial p}\frac{\partial \psi_o}{\partial q} \right] \int \psi_1 dp. \quad (12)$$

A dispersion relation for ω can then be drived from Eq. (12). Depending on the beam intensity and distribution as given by ψ_o, the imaginary part of ω may be positive, the perturbation will then grow exponentially. In general, a non-zero threshold intensity results from Landau damping. Only above the threshold is the particular type of perturbation unstable for the particular stationary distribution. The initial growths of these coherent instabilities are always exponential with growth rates given by the imaginary part of ω.

3. Development of instability

To study this, one must solve the full Vlasov equation. This can be done in general only with the help of a computer. The results of such computer studies may be summarized as follows:

"An initial stable and stationary distribution will remain unchanged in time. This is shown diagramatically in Fig. 1A."

"An initial stable but non-stationary distribution will undergo a 'damped oscillation' until it settles down to a stable and stationary distribution at large time. This is shown in Fig. 1B."

"For an initial stationary but unstable (above threshold) distribution the beam parameter (e.g. beam size) P governing the instability will grow until the distribution settles down to a stationary and stable (below threshold) one with a larger final value of P given by the overshoot formula due to Dory/4/

$$\left(P_{final}\right) \times \left(P_{initial}\right) = \left(P_{threshold}\right)^2. \quad (13)$$

Furthermore, the final stationary distribution may be different from the initial one. This is shown diagramatically in Fig. 1C."

"An initial non-stationary and unstable distribution grows in a manner similar to a stationary and unstable one, except it is 'oscillatory' from the beginning as shown in Fig. 1D."

In principle, all conerent instabilities can be damped by the use of a feedback system. The feedback system in effect modifies the impedance Z of the beam surrounding and raises the threshold of instability above the requirement given by the beam parameters. More directly, the impedance can be altered by physically modifying the electromagnetic characteristics of the beam enclosure.

Up until recently, a great deal of effort, both theoretical and experimental, was devoted to the discovery, the understanding, and the "cure" of dynamic instabilities. We now feel that most of the important coherent instabilities are understood and that the effectiveness of the "cures" is limited only by technology. With the application of superconducting magnets to the largest proton synchrotrons, an effort devoted to a more detailed re-examination of single particle dynamics seems to be most urgently needed. Perhaps with the much brighter beams made available by stochastic and electron cooling schemes, new high-current phenomena will come into view.

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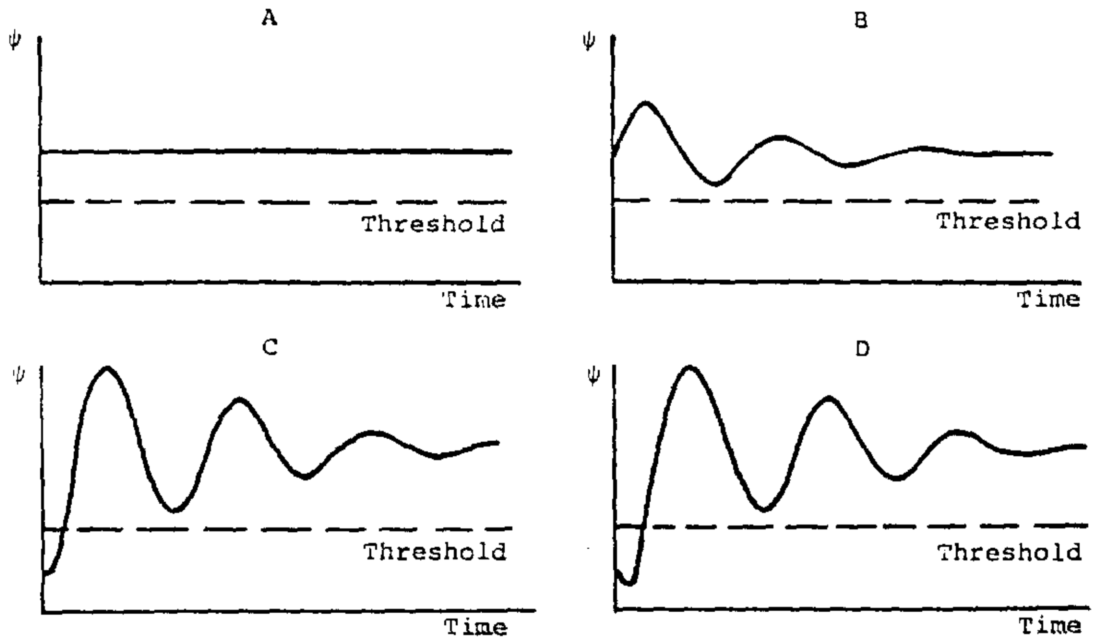

Figure 1. Idealized diagramatic representation of the time behavior of ψ

A. Initial ψ stationary and stable
B. Initial ψ non-stationary but stable
C. Initial ψ stationary but unstable
D. Initial ψ non-stationary and unstable